STATISTICAL INFERENCE FOR MEAN FUNCTION OF LONGITUDINAL IMAGING DATA OVER COMPLICATED DOMAINS

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Supplementary Materials

In this supplement, Section A and B provide some basic lemmas and proofs of theorems respectively. Section C reports additional simulation results.

Throughout this supplementary document, $\mathcal{O}_p$ (or $\scriptstyle\mathcal{O}_p$) denotes a sequence of random variables of certain order in probability. For instance, $\scriptstyle\mathcal{O}_p(n^{-1/2})$ means a smaller order than $n^{-1/2}$ in probability, and by $\mathcal{O}_{a.s.}$ (or $\scriptstyle\mathcal{O}_{a.s.}$) almost surely $\mathcal{O}$ (or $\scriptstyle\mathcal{O}$). In addition, $\mathcal{U}_p$ denotes a sequence of random functions which are $\mathcal{O}_p$ uniformly defined in the domain. For any vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{R}^n$, denote the norm $\|\mathbf{a}\|_r = (|a_1|^r + \cdots + |a_n|^r)^{1/r}$, $1 \leq r < +\infty$, $\|\mathbf{a}\|_\infty = \max(|a_1|, \ldots, |a_n|)$. For any matrix $\mathbf{A} = (a_{ij})_{i=1,j=1}^{m,n}$, denote its L_r norm as $\|\mathbf{A}\|_r = \max_{\mathbf{a}\in\mathcal{R}^n,\mathbf{a}\neq\mathbf{0}} \|\mathbf{A}\mathbf{a}\|_r \|\mathbf{a}\|_r^{-1}$, for $r < +\infty$ and $\|\mathbf{A}\|_r = \max_{1\leq i\leq m} \sum_{j=1}^n |a_{ij}|$, for $r = \infty$. For any random variable $\mathbf{X}$, if it is L^p-integrable, denotes its L^p norm as $\|\mathbf{X}\|_p = (\mathbb{E}\,|\mathbf{X}|^p)^{1/p}$.

A. Preliminaries

In order to investigate the estimation structure in greater depth, we decompose the estimation error $\widehat{\eta}_t(\boldsymbol{x}) - \eta_t(\boldsymbol{x})$ into three terms convenient. As the same sequence order of equation (2.5), denote the data vector $\boldsymbol{m} = \left(\{m(\boldsymbol{x}_{ij})\}_{i=1,j=1}^{M,N_i}\right)^\top$, $\boldsymbol{e}_t = \left(\{\sigma(\boldsymbol{x}_{ij})\varepsilon_{t,ij}\}_{i=1,j=1}^{M,N_i}\right)^\top$, $\boldsymbol{\phi}_k = \left(\{\phi_k(\boldsymbol{x}_{ij})\}_{i=1,j=1}^{M,N_i}\right)^\top$ and $\mathbf{R}_t = \sum_{k=1}^{\infty} \xi_{tk}\boldsymbol{\phi}_k$. Then the estimator $\widehat{\eta}_t(\boldsymbol{x})$ can be decomposed into there terms:

$$\widehat{\eta}_t(\boldsymbol{x}) = \widetilde{m}(\boldsymbol{x}) + \widetilde{\xi}_t(\boldsymbol{x}) + \widetilde{e}_t(\boldsymbol{x}), \tag{S.1}$$

where $\widetilde{m}(\boldsymbol{x})$, $\widetilde{\xi}_t(\boldsymbol{x})$, $\widetilde{e}_t(\boldsymbol{x})$ are the solutions of (2.6) with $Y_{t,ij}$ replaced by $m(\boldsymbol{x}_{ij})$, $R_t(\boldsymbol{x}_{ij})$, $\sigma(\boldsymbol{x}_{ij})\varepsilon_{t,ij}$ respectively, i.e. $\widetilde{m}(\boldsymbol{x}) = \widetilde{\mathbf{B}}(\boldsymbol{x})^\top \left(\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}\right)^{-1} \widetilde{\mathbf{X}}^\top\mathbf{m}$, $\widetilde{\xi}_t(\boldsymbol{x}) = \widetilde{\mathbf{B}}(\boldsymbol{x})^\top \left(\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}\right)^{-1} \widetilde{\mathbf{X}}^\top\mathbf{R}_t$ and $\widetilde{e}_t(\boldsymbol{x}) = \widetilde{\mathbf{B}}(\boldsymbol{x})^\top \left(\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}\right)^{-1} \widetilde{\mathbf{X}}^\top\mathbf{e}_t$.

For any L^2 integrable functions $\phi(\boldsymbol{x})$ and $\varphi(\boldsymbol{x})$ defined on Ω, take $\langle\phi,\varphi\rangle = \int_\Omega \phi(\boldsymbol{x})\varphi(\boldsymbol{x})d\boldsymbol{x}$ as their theoretical inner product and $\langle\phi,\varphi\rangle_{2,N} = N^{-1}\sum_{i=1}^{N}\phi(\boldsymbol{x}_i)\varphi(\boldsymbol{x}_i)$ as their empirical inner product, with regular L_2 norm $\|\phi\|_{L_2}^2 = \langle\phi,\phi\rangle$ and empirical norm $\|\phi\|_{2,N}^2 = \langle\phi,\phi\rangle_{2,N}$.

Recall the set of transformed Bernstein basis polynomials $\left\{\widetilde{B}_\ell(\boldsymbol{x})\right\}_{\ell=1}^{q}$ and $\widetilde{\mathbf{B}}(\boldsymbol{x}) = \mathbf{Q}_2^\top\mathbf{B}(\boldsymbol{x})$, $\widetilde{\mathbf{X}} = \mathbf{X}\mathbf{Q}_2$ defined in Section 2.2, denote by

$$\boldsymbol{\Gamma}_{N,0} = N^{-1}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}} = \left(\left\langle\widetilde{B}_\ell(\boldsymbol{x}), \widetilde{B}_{\ell'}(\boldsymbol{x})\right\rangle_{2,N}\right)_{\ell,\ell'=1}^{q}, \quad \mathbf{V} = \mathbf{Q}_2\boldsymbol{\Gamma}_{N,0}^{-1}\mathbf{Q}_2^\top,$$

two symmetric positive definite matrices.

Lemma A.1. (Lemma B.6 of Wang et al. (2020)) *Suppose that* $\triangle$ *is a* π*-quasi-uniform triangulation, if* $N^{1/2}|\triangle| \to \infty$ *as* $N \to \infty$*, then there exists constants* $0 < c < C < \infty$*, such that with probability approaching* 1 *as* $N \to \infty$*,* $n \to \infty$*, one has*

$$c|\triangle|^2 \leq \lambda_{min}(\mathbf{\Gamma}_{N,0}) \leq \lambda_{max}(\mathbf{\Gamma}_{N,0}) \leq C|\triangle|^2$$

Lemma A.2. (Theorem 10.10 of Lai and Schumaker (2007)) *Suppose that* $\triangle$ *is a* π*-quasi-uniform triangulation of a polygonal domain* Ω*, and* $g(\cdot) \in \mathcal{W}^{d+1,\infty}(\Omega)$*. For bi-integer* (a_1, a_2) *with* $0 \leq a_1 + a_2 \leq d$*, there exists a spline* $g^* \in S_d^r(\triangle)$ $(d \geq 3r+2)$ *such that* $\left\|\nabla_{z_1}^{a_1}\nabla_{z_2}^{a_2}(g - g^*)\right\|_\infty \leq C|\triangle|^{d+1-a_1-a_2}|g|_{d+1,\infty}$*, where* C *is a constant depending on* d*,* r *and the shape parameter* π*.*

Lemma A.3. (Theorem 2.6.7 of Csőrgő and Révész (1981)) *Suppose that* ξ_i*,* $1 \leq i \leq n$ *are iid with* $\mathbb{E}(\xi_1) = 0$*,* $\mathbb{E}(\xi_1^2) = 1$ *and* $H(x) > 0$ $(x \geq 0)$ *is an increasing continuous function such that* $x^{-2-\gamma}H(x)$ *is increasing for some* $\gamma > 0$ *and* $x^{-1}\log H(x)$ *is decreasing with* $\mathbb{E}H(|\xi_1|) < \infty$*. Then there exist constants* C_1*,* C_2*,* $a > 0$ *which depend only on the distribution of* ξ_1 *and a sequence of Brownian motions* $\{W_n(m)\}_{n=1}^\infty$*, such that for any* $\{x_n\}_{n=1}^\infty$ *satisfying* $H^{-1}(n) < x_n < C_1(n\log n)^{1/2}$ *and* $S_m = \sum_{i=1}^m \xi_i$*, then* $\mathbb{P}\{\max_{1\leq m\leq n}|S_m - W_n(m)| > x_n\} \leq C_2 n\{H(ax_n)\}^{-1}$*.*

Lemma A.4. (Theorem 1.5.4 of van der Vaart (1998)) T *is a arbitrary set. Let*

$X_\alpha : \Omega_\alpha \to \ell^\infty(T)$ be arbitrary. Then X_α converges weakly to a tight limit if and only if X_α is asymptotically tight and the marginals $(X_\alpha(t_1), \ldots, X_\alpha(t_k))$ converge weakly to a limit for every finite subset $t_1, \ldots, t_k$ of T. If X_α is asymptotically tight and its marginals converge weakly to the marginals $(X_\alpha(t_1), \ldots, X_\alpha(t_k))$ of a stochastic process X, then there is a version of X with uniformly bounded sample paths and $X_\alpha \xrightarrow{d} X$.

Lemma A.5. (Theorem 1.5.6 of van der Vaart (1998)) *A net $X_\alpha : \Omega_\alpha \to \ell^\infty(T)$ is asymptotically tight if and only if X_α is asymptotically tight in $\mathbb{R}$ for every t and s, for all $\varepsilon, \eta > 0$, there exists a finite partition $T = \cup_{i=1}^k T_i$ such that*

$$\limsup_{\alpha \to \infty} \mathbb{P}\left(\sup_{1 \le i \le k} \sup_{s,t \in T_i} |X_\alpha(s) - X_\alpha(t)| > \varepsilon \right) < \eta. \tag{S.2}$$

Lemma A.6. *For a π-quasi-uniform triangulation $\triangle$, if $N^{1/2} |\triangle| \to \infty$ as $N \to \infty$, then there exists constants $0 < c < C < \infty$, such that with probability approaching 1 as $N \to \infty$, $n \to \infty$, one has*

$$c\,|\triangle|^{-2} \le \lambda_{min}(\mathbf{V}) \le \lambda_{max}(\mathbf{V}) \le C\,|\triangle|^{-2}$$

Proof. For any q-dimensional vector $\boldsymbol{\theta}$, one has $\boldsymbol{\theta}^\top \mathbf{V} \boldsymbol{\theta} = \boldsymbol{\theta}^\top \mathbf{Q}_2 \boldsymbol{\Gamma}_{N,0}^{-1} \mathbf{Q}_2^\top \boldsymbol{\theta}$.

According to Lemma A.1, there exists

$$c\,|\triangle|^{-2}\left\|\mathbf{Q}_2^\top\boldsymbol{\theta}\right\|_2^2 \le \boldsymbol{\theta}^\top\mathbf{Q}_2\boldsymbol{\Gamma}_{N,0}^{-1}\mathbf{Q}_2^\top\boldsymbol{\theta} \le C\,|\triangle|^{-2}\left\|\mathbf{Q}_2^\top\boldsymbol{\theta}\right\|_2^2$$

Note that $\left\|\mathbf{Q}_2^\top\boldsymbol{\theta}\right\|_2^2 = \boldsymbol{\theta}^\top\mathbf{Q}_2\mathbf{Q}_2^\top\boldsymbol{\theta}$ and the eigenvalues of $\mathbf{Q}_2\mathbf{Q}_2^\top$ are either 0 or 1, thus $\left\|\mathbf{Q}_2^\top\boldsymbol{\theta}\right\|_2^2 \le \boldsymbol{\theta}^\top\boldsymbol{\theta}$, which leads to

$$c\,|\triangle|^{-2}\left\|\boldsymbol{\theta}\right\|_2^2 \le \boldsymbol{\theta}^\top\mathbf{Q}_2\boldsymbol{\Gamma}_{N,0}^{-1}\mathbf{Q}_2^\top\boldsymbol{\theta} \le C\,|\triangle|^{-2}\left\|\boldsymbol{\theta}\right\|_2^2.$$

Hence $c\,|\triangle|^{-2} \le \lambda_{min}(\mathbf{V}) \le \lambda_{max}(\mathbf{V}) \le C\,|\triangle|^{-2}$.

Lemma A.7. *For any Bernstein basis polynomials $B_l(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$ of degree $d \ge 0$, we have*

$$\sum_{i=1}^{M}\sum_{j=1}^{N_i} B_\ell(\boldsymbol{x}_{ij}) = \mathcal{O}\left(N|\triangle|^2\right), \qquad \forall \ell = 1,\dots,p, \tag{S.3}$$

$$\sum_{\ell=1}^{p} B_\ell\left(\boldsymbol{x}_{ij}\right) = \mathcal{O}(1), \qquad \forall i = 1,\dots M,\ j = 1,\dots,N_i, \tag{S.4}$$

$$\max_{1\le l,l'\le p}\ \max_{1\le i\le M, 1\le j\le N_i}\ \sup_{\boldsymbol{x}\in T_h}\left|B_\ell(\boldsymbol{x}_{ij})B_{\ell'}(\boldsymbol{x}_{ij}) - B_\ell(\boldsymbol{x})B_{\ell'}(\boldsymbol{x})\right| = \mathcal{O}\left(N^{-1/2}\,|\triangle|^{-1}\right), \tag{S.5}$$

where $T_h \in \triangle$ is the one which contains $\boldsymbol{x}_{ij}$.

Proof. It is trivial that (S.3) holds. For any fixed $\boldsymbol{x}_{ij}$, $i = 1,\dots M$, $j =$

$1, \ldots, N_i$, assume that $T_h \in \triangle$ is the triangle that contains $\boldsymbol{x}_{ij}$. Note that there are $d^* = (d+1)(d+2)/2$ Bernstein basis polynomials on each triangle, then

$$\sum_{\ell=1}^{p} B_\ell(\boldsymbol{x}_{ij}) = \sum_{\{\ell:\lceil \ell/d^* \rceil = h\}} B_l(\boldsymbol{x}_{ij}) \leq (d+1)(d+2)/2 = \mathcal{O}(1)$$

Denote by $\omega(f,h) = \max\left\{|f(x,y) - f(\tilde{x},\tilde{y})| : (x,y),(\tilde{x},\tilde{y}) \in T, |x-\tilde{x}|^2 + |y-\tilde{y}|^2 \leq h^2\right\}$ the modulus of continuity of f relative to the triangle T. Since for any $\ell = 1, \ldots, p$, $B_\ell(\cdot) \in C^1(T)$, then $\omega\left(B_\ell, N^{-1/2}\right) \leq N^{-1/2}|B_\ell|_{1,\infty,T} \leq CN^{-1/2}\left|\triangle\right|^{-1}$, thus

$$\begin{aligned}
&|B_\ell(\boldsymbol{x}_{ij})B_{\ell'}(\boldsymbol{x}_{ij}) - B_\ell(\boldsymbol{x})B_{\ell'}(\boldsymbol{x})| \\
&= |B_\ell(\boldsymbol{x}_{ij})B_{\ell'}(\boldsymbol{x}_{ij}) - B_\ell(\boldsymbol{x})B_{\ell'}(\boldsymbol{x}_{ij}) + B_\ell(\boldsymbol{x})B_{\ell'}(\boldsymbol{x}_{ij}) - B_\ell(\boldsymbol{x})B_{\ell'}(\boldsymbol{x})| \\
&= |B_{\ell'}(\boldsymbol{x}_{ij})|\,|B_\ell(\boldsymbol{x}_{ij}) - B_\ell(\boldsymbol{x})| + |B_\ell(\boldsymbol{x})|\,|B_{\ell'}(\boldsymbol{x}_{ij}) - B_{\ell'}(\boldsymbol{x})| \\
&= |B_{\ell'}(\boldsymbol{x}_{ij})|\,\omega\left(B_\ell, N^{-1/2}\right) + |B_\ell(\boldsymbol{x})|\,\omega\left(B_{\ell'}, N^{-1/2}\right) \\
&= \mathcal{O}\left(N^{-1/2}\left|\triangle\right|^{-1}\right).
\end{aligned}$$

The proof is completed. □

For any function $\phi \in C(\Omega)$, denote the vector $\boldsymbol{\phi} = (\phi(\boldsymbol{x}_{ij}))^\top$ as the order of (2.5) and the function $\widetilde{\phi}(\boldsymbol{x}) = \widetilde{\mathbf{B}}^\top(\boldsymbol{x})(\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}})^{-1}\widetilde{\mathbf{X}}^\top\boldsymbol{\phi}$.

Lemma A.8. *There exists $c_d \in (0,\infty)$ such that when n is large enough, $\|\widetilde{\phi}\|_{\infty,\Omega} \leq c_d\|\phi\|_{\infty,\Omega}$ for any $\phi \in C(\Omega)$. Furthermore, if $\phi \in \mathcal{W}^{d+1,\infty}(\Omega)$ for some $\mu \in (0,1]$,* *then there exists* $\widetilde{C}_{d,r}$*, such that*

$$\|\widetilde{\phi} - \phi\|_{\infty,\Omega} \leq \widetilde{C}_{d,r}\, |\phi|_{d+1,\infty,\Omega}\, |\triangle|^{d+1}.$$

Proof. Note that for any $\boldsymbol{x} \in \Omega$, at most $(d+1)(d+2)/2$ numbers of $B_1(\boldsymbol{x}), \cdots, B_p(\boldsymbol{x})$ are between 0 and 1, others being 0 , so

$$\begin{aligned}
\left\|\widetilde{\phi}\right\|_{\infty,\Omega} &= \left\|N^{-1}\mathbf{B}^\top(\boldsymbol{x})\mathbf{V}\mathbf{X}^\top\boldsymbol{\phi}\right\|_{\infty,\Omega} \\
&\leq \frac{(d+1)(d+2)}{2}N^{-1}\left\|\mathbf{V}\mathbf{X}^\top\boldsymbol{\phi}\right\|_\infty \\
&\leq C\frac{(d+1)(d+2)}{2N}|\triangle|^{-2}\,\|\phi\|_{\infty,\Omega}\left\|\mathbf{X}^\top\mathbf{1}_N\right\|_\infty,
\end{aligned}$$

in which $\mathbf{1}_N = (1,\dots,1)^\top$ is a N-dimensional constant vector. Clearly, (S.3) ensures that

$$\left\|X^T\mathbf{1}_N\right\|_\infty = \max_{1\leq l\leq p}\sum_{i=1}^{M}\sum_{j=1}^{N_i} B_l(\boldsymbol{x}_{ij}) \leq CN|\triangle|^2,$$

which implies $\|\widetilde{\phi}\|_{\infty,\Omega} \leq c_d\|\phi\|_{\infty,\Omega}$.

Now if $\phi \in \mathcal{W}^{d+1,\infty}(\Omega)$, let $g \in S_d^r(\triangle)$ be such that $\|g-\phi\|_\infty \leq C_{d,r}\,|\triangle|^{d+1}\,|\phi|_{d+1,\infty,\Omega}$ according to Lemma A.2, then $\widetilde{g} \equiv g$ as $g \in S_d^r(\triangle)$, hence

$$\begin{aligned}
\|\widetilde{\phi} - \phi\|_{\infty,\Omega} &= \|\widetilde{\phi} - \widetilde{g}\|_{\infty,\Omega} + \|\phi - g\|_{\infty,\Omega} \\
&\leq (c_d+1)\|\phi - g\|_{\infty,\Omega} \leq \widetilde{C}_{d,r}\,|\phi|_{d+1,\infty,\Omega}\,|\triangle|^{d+1}
\end{aligned}$$

The proof is completed.

Lemma A.9. *For $n > 2$, $a > 2$, $W_i \sim \mathrm{N}\left(0, \sigma_i^2\right), \sigma_i > 0, i = 1, \ldots, n$*

$$\mathbb{P}\left(\max_{1 \le i \le n} |W_i/\sigma_i| > a\sqrt{\log n}\right) < \sqrt{\frac{2}{\pi}} n^{1-a^2/2}. \tag{S.6}$$

As $n \to \infty$, $\left(\max_{1\le i \le n} |W_i|\right) / \left(\max_{1 \le i \le n} \sigma_i\right) \le \max_{1\le i\le n} |W_i/\sigma_i| = \mathcal{O}_{a.s.}\left(\sqrt{\log n}\right)$.

Proof. Note that

$$\begin{aligned}\mathbb{P}\left(\max_{1\le i\le n}\left|\frac{W_i}{\sigma_i}\right| > a\sqrt{\log n}\right) &\le \sum_{i=1}^{n} \mathbb{P}\left(\left|\frac{W_i}{\sigma_i}\right| > a\sqrt{\log n}\right) \le 2n\left\{1 - \Phi\left(a\sqrt{\log n}\right)\right\} \\ &< 2n\frac{\phi\left(a\sqrt{\log n}\right)}{a\sqrt{\log n}} \le 2n\phi\left(a\sqrt{\log n}\right) = \sqrt{\frac{2}{\pi}} n^{1-a^2/2},\end{aligned}$$

for $n > 2$, $a > 2$, which proves (S.6). The lemma follows by applying Borel-Cantelli Lemma with choice of $a > 2$.

Lemma A.10. *Assumption (A5) holds under Assumptions (A3), (A4) and (A5′).*

Proof. Under Assumption (A5′), $\mathbb{E}\left|\zeta_{tk}\right|^{r_1} < \infty$, $r_1 > 4 + 2\alpha$, $\mathbb{E}\left|\varepsilon_{t,ij}\right|^{r_2} < \infty$, $r_2\omega > 2 + \theta$, where α is defined in Assumption (A4) and θ is defined in Assumption (A3), so there exists some $\beta_1 \in (0, 1/2)$, such that $r_1 > (2 + \alpha)/\beta_1$.

Let $H(x) = x^{r_1}$. Lemma A.3 entails that there exist constants c_{1k} and a_k depending on the distribution of ζ_{tk}, such that for $x_n = (n + I_n)^{\beta_1}$, $(n+I_n)/H\left(a_k x_n\right) =$

$a_k^{-r_1}(n+I_n)^{1-r_1\beta_1}$ and iid $\mathrm{N}(0,1)$ variables $Z_{tk,\zeta}$,

$$\mathbb{P}\left\{\max_{-I_n+1\le\tau\le n}\left|\sum_{t=-I_n+1}^{\tau}\zeta_{tk}-\sum_{t=-I_n+1}^{\tau}Z_{tk,\zeta}\right|>(n+I_n)^{\beta_1}\right\}<c_{1k}a_k^{-r_1}(n+I_n)^{1-r_1\beta_1},$$

Since there are only a finite number of distinct distributions for $\{\zeta_{tk}\}_{t=-I_n+1,k=1}^{n,k_n}$ by Assumption (A5′), there exists a common $c_1>0$, such that

$$\max_{1\le k\le k_n}\mathbb{P}\left\{\max_{-I_n+1\le\tau\le n}\left|\sum_{t=-I_n+1}^{\tau}\zeta_{t,k}-\sum_{t=-I_n+1}^{\tau}Z_{tk,\zeta}\right|>(n+I_n)^{\beta_1}\right\}<c_1(n+I_n)^{1-r_1\beta_1}.$$

Since $I_n\asymp n^{\iota},0<\iota<1$, by definition, then C_r inequality leads to that $(n+I_n)^{\beta_1}\le C_0n^{\beta_1}$ for some constant C_0. Because $1-r_1\beta_1<0$, it is clear that $(n+I_n)^{1-r_1\beta_1}<n^{1-r_1\beta_1}$. Thus one has

$$\max_{1\le k\le k_n}\mathbb{P}\left\{\max_{-I_n+1\le\tau\le n}\left|\sum_{t=-I_n+1}^{\tau}\zeta_{t,k}-\sum_{t=-I_n+1}^{\tau}Z_{tk,\zeta}\right|>C_0n^{\beta_1}\right\}<c_1n^{1-r_1\beta_1}.$$

Recalling that $r_1>(2+\alpha)/\beta_1$, one can let $\gamma_1=r_1\beta_1-1-\alpha>1$, and there exists a $C_1>0$ such that

$$\mathbb{P}\left\{\max_{1\le k\le k_n}\max_{-I_n+1\le\tau\le n}\left|\sum_{t=-I_n+1}^{\tau}\zeta_{t,k}-\sum_{t=-I_n+1}^{\tau}Z_{tk,\zeta}\right|>C_0n^{\beta_1}\right\}<k_nc_1n^{1-r_1\beta_1}\le C_1n^{-\gamma_1}.$$

Similarly, under Assumption (A5′), let $H(x)=x^{r_2}$, Lemma A.3 entails that there exists constants c_2 and b depending on the distribution of ε_{ij}, such that for $x_N = N^{\beta_2}$, $N/H(bx_N) = b^{-r_2}N^{1-r_2\beta_2}$ and iid standard normal random variables $\{Z_{t,ij,\varepsilon}\}_{t=1,i=1,j=1}^{n,M,N_i}$ such that

$$\max_{1\le t\le n} \mathbb{P}\left\{\max_{1\le\tau\le N}\left|\sum_{k=1}^{\tau}\varepsilon_{t,f_1(k)f_2(k)} - \sum_{k=1}^{\tau} Z_{t,f_1(k)f_2(k),\varepsilon}\right| > N^{\beta_2}\right\} \le c_2 b^{-r_2}N^{1-r_2\beta_2}$$

Assumption (A3) states that $n = \mathcal{O}\left(N^{\theta}\right)$, so there is a $C_2 > 0$ such that

$$\mathbb{P}\left\{\max_{1\le t\le n}\max_{1\le\tau\le N}\left|\sum_{k=1}^{\tau}\varepsilon_{t,f_1(k)f_2(k)} - \sum_{k=1}^{\tau} Z_{t,f_1(k)f_2(k),\varepsilon}\right| > N^{\beta_2}\right\} \le C_2 N^{\theta+1-r_2\beta_2}$$

Note that $r_2\omega > 2+\theta$, one can choose $\beta_2 \in (0,\omega)$, such that $r_2\beta_2 > 2+\theta$, which ensures that $\gamma_2 = \beta_2 r_2 - 1 - \theta > 1$ and Assumption (A5) follows.

The lemma holds consequently.

Lemma A.11. *Under Assumptions (A5) and (A5′), as $n \to \infty$, there are constants $C_3, C_4 \in (0,+\infty)$, $\gamma_3 \in (1,+\infty)$, $\beta_3 \in (0,1/2)$ and a series of* $\mathrm{N}(0,1)$ *variables $Z_{tk,\xi} = \sum_{t'=0}^{\infty} a_{t'k}Z_{t-t',k,\zeta}$, $t = 1,\ldots,n$, $k = 1,\ldots,k_n$, with* $\mathrm{Cov}\left(Z_{jk,\xi}, Z_{j+h,k,\xi}\right) = \sum_{m=0}^{\infty} a_{mk}a_{m+h,k}$, *$1 \le j \le n$, $1 \le h \le n-j$, such that*

$$\mathbb{P}\left\{\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\xi_{tk} - \sum_{t=1}^{\tau} Z_{tk,\xi}\right| > C_3 n^{\beta_3}\right\} < C_4 n^{-\gamma_3}. \tag{S.7}$$

Proof. Since $\sum_{t=0}^{\infty} a_{tk}^2 = 1$ and $|a_{tk}| < C_a t^{\rho_a}$, for $t = 0,\ldots,n$, $k = 1,\ldots,k_n$, together with $I_n \asymp n^\iota$, then $(t' + I_n)^{\rho_a} \le C\,(t' I_n)^{\rho_a/2}$ for $t' \ge 1$ and some constant C. There also exists a constant $M > 0$, such that $\sum_{t=0}^{I_n} |a_{tk}| < M$. It is clear that

$$\xi_{tk} = \sum_{t'=0}^{I_n} a_{t'k}\zeta_{t-t',k} + \sum_{t'=I_n+1}^{\infty} a_{t'k}\zeta_{t-t',k},$$

$$\left|\xi_{tk} - \sum_{t'=0}^{I_n} a_{t'k}\zeta_{t-t',k}\right| \le \sum_{t'=I_n+1}^{\infty} C_a t'^{\rho_a} \left|\zeta_{t-t',k}\right|,$$

$$\left|\xi_{tk} - \sum_{t'=0}^{I_n} a_{t'k}\zeta_{t-t',k}\right| \le C n^{\rho_a \iota} \sum_{t'=1}^{\infty} t'^{\rho_a/2} \left|\zeta_{t-I_n-t',k}\right|.$$

Hence,

$$\max_{1\le k\le k_n} \max_{1\le\tau\le n} \left|\sum_{t=1}^{\tau} \xi_{tk} - \sum_{t=1}^{\tau}\sum_{t'=0}^{I_n} a_{t'k}\zeta_{t-t',k}\right| \le \max_{1\le k\le k_n} \max_{1\le t\le n} C n^{\rho_a\iota/2+1} \sum_{t'=1}^{\infty} t'^{\rho_a/2} \left|\zeta_{t-I_n-t',k}\right|$$

Denote $W_{tk} = \sum_{t'=1}^{\infty} t'^{\rho_a} |\zeta_{t-I_n-t',k}|$, by noticing that $\sup_{t,k} \mathbb{E}\left|\zeta_{t,k}\right|^{r_1} < \infty$, where $r_1 > 4 + 2\alpha$,

$$\left\|W_{tk}\right\|_{r_1} \le \sum_{t'=1}^{\infty} t'^{\rho_a/2} \left\|\zeta_{t-I_n-t',k}\right\|_{r_1} < \infty.$$

Therefore, $\mathbb{E}W_{tk}^{r_1} < K$ for some $K > 0$, $t = 1, \ldots, n$, $k = 1, \ldots, k_n$. Note that $k_n = \mathcal{O}\left(n^\alpha\right)$ in Assumption (A4), thus

$$\mathbb{P}\left(C n^{\rho_a\iota/2+1} \max_{1\le k\le k_n} \max_{1\le t\le n} W_{tk} > M n^{\beta_3}\right) < n k_n \frac{C^{r_1} K}{M^{r_1}} n^{-(\beta_3-1) r_1} < \frac{C^{r_1} K}{M^{r_1}} n^{-(\beta_3-\rho_a\iota/2-1) r_1+1+\alpha}.$$

So,

$$\mathbb{P}\left[\max_{1\leq k\leq k_n}\max_{1\leq\tau\leq n}\left|\sum_{t=1}^{\tau}\xi_{tk}-\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\zeta_{t-t',k}\right|>Mn^{\beta_3}\right]<\frac{C^{r_1}K}{M^{r_1}}n^{-(\beta_3-\rho_a\iota/2-1)r_1+1+\alpha}.$$

Next, define $U_{tk}=\sum_{t'=I_n+1}^{\infty}a_{t'k}Z_{t-t',k,\zeta}$, then $U_{tk}\sim N\left(0,\sum_{t'=I_n+1}^{\infty}a_{t'k}^2\right)$, $k=1,\ldots,k_n$. It is obvious that

$$\max_{1\leq k\leq k_n}\max_{1\leq\tau\leq n}\left|\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty}a_{t'k}Z_{t-t',k,\zeta}\right|\leq n\max_{1\leq k\leq k_n}\max_{1\leq t\leq n}\left|U_{tk}\right|.$$

Note that $\sum_{t'=I_n+1}^{\infty}a_{t'k}^2<Cn^{(2\rho_a+1)\iota}$ for some $C>0$, $k=1,\ldots,k_n$ and $k_n=\mathcal{O}\left(n^{\alpha}\right)$ for some $\alpha>0$, one has

$$\mathbb{P}\left(n\max_{1\leq k\leq k_n}\max_{1\leq t\leq n}\left|U_{tk}\right|>Mn^{\beta_3}\right)<nk_n\frac{Cn^{(2\rho_a+1)\iota}}{M^2\left(n^{\beta_3-1}\right)^2}<\frac{C}{M^2}n^{(2\rho_a+1)\iota-2\beta_3+\alpha+3},$$

which leads to

$$\mathbb{P}\left[\max_{1\leq k\leq k_n}\max_{1\leq\tau\leq n}\left|\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty}a_{t'k}Z_{t-t',k,\zeta}\right|>Mn^{\beta_3}\right]<\frac{C}{M^2}n^{(2\rho_a+1)\iota-2\beta_3+\alpha+3}.$$

Now Assumption (A5) entails that for $0\leq t'\leq I_n$, $1\leq t\leq n$, $-I_n+1\leq t-t'\leq n$

$$\mathbb{P}\left\{\max_{1\leq k\leq k_n}\max_{-I_n+1\leq\tau\leq n}\left|\sum_{t=-I_n+1}^{\tau}\zeta_{tk}-\sum_{t=-I_n+1}^{\tau}Z_{tk,\zeta}\right|>C_0n^{\beta_3}\right\}<C_1n^{-\gamma_1}.$$

Then,

$$
\begin{aligned}
&\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}(\zeta_{t-t',k}-Z_{t-t',k,\zeta})\right|>2MC_0n^{\beta_3}\right]\\
&=\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t'=0}^{I_n}a_{t'k}\sum_{t=1}^{\tau}(\zeta_{t-t',k}-Z_{t-t',k,\zeta})\right|>2MC_0n^{\beta_3}\right]\\
&\le\mathbb{P}\left[\max_{1\le k\le k_n}\left\{\sum_{t'=0}^{I_n}|a_{t'k}|\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\zeta_{t-t',k}-\sum_{t=1}^{\tau}Z_{t-t',k,\zeta}\right|\right\}>2MC_0n^{\beta_3}\right]\\
&\le\mathbb{P}\left[M\max_{1\le k\le k_n}\max_{1\le\tau\le n}\max_{0\le t'\le I_n}\left|\sum_{t=1}^{\tau}\zeta_{t-t',k}-\sum_{t=1}^{\tau}Z_{t-t',k,\zeta}\right|>2MC_0n^{\beta_3}\right]\\
&\le\mathbb{P}\left\{2\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=-I_n+1}^{\tau}\zeta_{tk}-\sum_{t=-I_n+1}^{\tau}Z_{tk,\zeta}\right|>2C_0n^{\beta_3}\right\}<C_1n^{-\gamma_1}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\xi_{tk}-\sum_{t=1}^{\tau}\left(\sum_{t'=0}^{\infty}a_{t'k}Z_{t-t',k,\zeta}\right)\right|>4MC_0n^{\beta_3}\right]\\
&=\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\xi_{tk}-\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\zeta_{t-t',k}+\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\zeta_{t-t',k}-\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}Z_{t-t',k,\zeta}\right.\right.\\
&\qquad\left.\left.-\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty}a_{t'k}Z_{t-t',k,\zeta}\right|>4MC_0n^{\beta_3}\right]\\
&\le\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left\{\left|\sum_{t=1}^{\tau}\xi_{tk}-\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\zeta_{t-t',k}\right|+\left|\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\zeta_{t-t',k}-\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}Z_{t-t',k,\zeta}\right|\right.\right.\\
&\qquad\left.\left.+\left|\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty}a_{t'k}Z_{t-t',k,\zeta}\right|\right\}>4MC_0n^{\beta_3}\right]\\
&\le\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\xi_{tk}-\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\zeta_{t-t',k}\right|>MC_0n^{\beta_3}\right]+\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n}a_{t'k}\zeta_{t-t',k}\right.\right.
\end{aligned}
$$

$$-\sum_{t=1}^{\tau}\sum_{t'=0}^{I_n} a_{t'k}Z_{t-t',k,\zeta}\right| > 2MC_0n^{\beta_3}\Bigg] + \mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\sum_{t'=I_n+1}^{\infty} a_{t'k}Z_{t-t',k,\zeta}\right| > MC_0n^{\beta_3}\right]$$

$$\le \frac{C^{r_1}K}{M^{r_1}}n^{-(\beta_3-\rho_a\iota/2-1)r_1+1+\alpha} + C_1n^{-\gamma_1} + \frac{C}{M^2}n^{(2\rho_a+1)\iota-2\beta_3+\alpha+3} < C_4n^{-\gamma_3}$$

Denote $C_3 = 4MC_0$ and $Z_{tk,\xi} = \sum_{t'=0}^{\infty} a_{t'k}Z_{t-t',k,\zeta}$, $t = 1,\ldots,n$, $k = 1,\ldots,k_n$, then $\{Z_{tk,\xi}\}_{t=1,k=1}^{n,k_n}$ are N$(0,1)$ variables and $\mathrm{Cov}\left(Z_{j,k,\xi}, Z_{j+h,k,\xi}\right) = \sum_{m=0}^{\infty} a_{mk}a_{m+h,k}$, $1 \le j \le n$, $1 \le h \le n-j$, thus

$$\mathbb{P}\left[\max_{1\le k\le k_n}\max_{1\le\tau\le n}\left|\sum_{t=1}^{\tau}\xi_{tk} - \sum_{t=1}^{\tau}Z_{tk,\xi}\right| > C_3n^{\beta_3}\right] < C_4n^{-\gamma_3}.$$

The proof is completed.

Lemma A.12. *Under Assumptions (A2), (A5) and (A6),*

$$\max_{1\le\ell\le p}\left|(nN)^{-1}\sum_{t=1}^{n}\sum_{i=1}^{M}\sum_{j=1}^{N_i} B_{\ell,p}(\boldsymbol{x}_{ij})\sigma\left(\boldsymbol{x}_{ij}\right)Z_{t,ij,\varepsilon}\right| = \mathcal{O}_{a.s.}(n^{-1/2}N^{-1/2}|\triangle|\log^{1/2}N).$$

PROOF. Note that $(nN)^{-1}\sum_{t=1}^{n}\sum_{i=1}^{M}\sum_{j=1}^{N_i} B_\ell(\boldsymbol{x}_{ij})\sigma\left(\boldsymbol{x}_{ij}\right)Z_{t,ij,\varepsilon} =$ $N^{-1}\sum_{i=1}^{M}\sum_{j=1}^{N_i} B_{\ell,p}(\boldsymbol{x}_{ij})\sigma\left(\boldsymbol{x}_{ij}\right)Z_{\cdot ij,\varepsilon}$, where $Z_{\cdot ij,\varepsilon} = n^{-1}\sum_{t=1}^{n} Z_{t,ij,\varepsilon}$, then apply Lemma A.9 to obtain the uniform bound for the zero mean Gaussian variables $N^{-1}\sum_{i=1}^{N} B_\ell(\boldsymbol{x}_{ij})\sigma\left(\boldsymbol{x}_{ij}\right)Z_{\cdot ij,\varepsilon}$, $1 \le \ell \le p$ with variance

$$\mathbb{E}\left\{N^{-1}\sum_{i=1}^{M}\sum_{j=1}^{N_i} B_{\ell,p}(\boldsymbol{x}_{ij})\sigma\left(\boldsymbol{x}_{ij}\right)Z_{\cdot ij,\varepsilon}\right\}^2 = n^{-1}N^{-2}\sum_{i=1}^{M}\sum_{j=1}^{N_i} B_{\ell,p}^2\left(\boldsymbol{x}_{ij}\right)\sigma^2\left(\boldsymbol{x}_{ij}\right)$$

$$= n^{-1}N^{-1}\left\|B_{\ell,p}\sigma\right\|_{2,N}^{2} \asymp |\triangle|^{2}N^{-1}n^{-1}.$$

It follows from Lemma A.9 that

$$\max_{1\le\ell\le p}\left|N^{-1}\sum_{i=1}^{M}\sum_{j=1}^{N_i}B_{\ell,p}(\boldsymbol{x}_{ij})\sigma\left(\boldsymbol{x}_{ij}\right)Z_{\cdot ij,\varepsilon}\right| = \mathcal{O}_{a.s.}\left\{n^{-1/2}N^{-1/2}|\triangle|\log^{1/2}p\right\}$$
$$= \mathcal{O}_{a.s.}\left(n^{-1/2}N^{-1/2}|\triangle|\log^{1/2}N\right), \quad \text{(S.8)}$$

where the last step follows from Assumption (A6) on the order of $|\triangle|$ relative to N. Thus the lemma holds.

Lemma A.13. *Under Assumptions (A2), (A5) and (A6), one has*

$$\sup_{\boldsymbol{x}\in\Omega} n^{-1}\left|\sum_{t=1}^{n}\widetilde{\boldsymbol{e}}_t(\boldsymbol{x})\right| = \mathcal{O}_{a.s}\left(n^{-1/2}N^{-1/2}|\triangle|^{-1}\log^{1/2}N + N^{\beta_2-1/2}|\triangle|^{-1} + N^{\beta_2-1}|\triangle|^{-2}\right)$$

Proof. According to Assumption (A5), it is trivial that

$$\max_{1\le t\le n}\max_{1\le\tau\le N}\left|N^{-1}\sum_{k=1}^{\tau}\left(\varepsilon_{t,f_1(k)f_2(k)} - Z_{t,f_1(k)f_2(k),\varepsilon}\right)\right| = \mathcal{O}_{a.s.}\left(N^{\beta_2-1}\right).$$

The bivariate spline satisfies that

$$\left|B_\ell\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) - B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\right| \le C|\triangle|^{-1}N^{-1/2}$$

uniformly over $1 \leq k \leq N$ and $1 \leq l \leq p$, while the Lipschitz continuity in Assumption (A2) ensures that

$$\left|\sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) - \sigma\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\right| \leq LN^{-1/2} \leq C|\triangle|^{-1}N^{-1/2}$$

uniformly over $1 \leq k \leq N$. Note that for $1 \leq \ell \leq p$, both $B_\ell(\cdot)$ and $\sigma(\cdot)$ are bounded on Ω, then

$$\begin{aligned}
&\left|B_\ell\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right)\sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) - B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\sigma\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\right| \\
=&\left|\left\{B_\ell\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) - B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right) + B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\right\}\sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right)\right. \\
&\left.-B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\sigma\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\right| \\
\leq&\left|B_\ell\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) - B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\right|\sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) \\
&+\left|\sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) - \sigma\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\right|B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right) \\
\leq&C|\triangle|^{-1}N^{-1/2}
\end{aligned}$$

Noting the support set of $B_\ell(\cdot)$, one obtains that

$$\begin{aligned}
&\sum_{k=1}^{N-1}\left|B_\ell\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right)\sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) - B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\sigma\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right)\right| \\
&\leq CN|\triangle|^2|\triangle|^{-1}N^{-1/2} \leq CN^{1/2}|\triangle|.
\end{aligned}$$

Hence, for $1 \le \ell \le p$, $1 \le t \le n$,

$$
\begin{aligned}
& N^{-1} \sum_{k=1}^{N} B_\ell\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) \sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right)\left(Z_{t,f_1(k)f_2(k),\varepsilon} - \varepsilon_{t,f_1(k)f_2(k)}\right) \\
= \; & N^{-1} \sum_{k=1}^{N-1} \Big\{ \big[B_\ell\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) \sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) \\
& - B_\ell\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right) \sigma\left(\boldsymbol{x}_{f_1(k+1)f_2(k+1)}\right) \big] \sum_{m=1}^{k} \left(Z_{t,f_1(m)f_2(m),\varepsilon} - \varepsilon_{t,f_1(m)f_2(m)}\right) \Big\} \\
& + N^{-1} B_\ell\left(\boldsymbol{x}_{f_1(N)f_2(N)}\right) \sigma\left(\boldsymbol{x}_{f_1(N)f_2(N)}\right) \sum_{k=1}^{N} \left(Z_{t,f_1(k)f_2(k),\varepsilon} - \varepsilon_{t,f_1(k)f_2(k)}\right) \\
\le \; & \left\{ \max_{1 \le t \le n} \max_{1 \le \tau \le N} \left| N^{-1} \sum_{k=1}^{\tau} \left(\varepsilon_{t,f_1(k)f_2(k)} - Z_{t,f_1(k)f_2(k),\varepsilon}\right) \right| \right\} \\
& \times \left(C N^{1/2} |\triangle|\right) + C \left| N^{-1} \sum_{k=1}^{N} \left(Z_{t,f_1(k)f_2(k),\varepsilon} - \varepsilon_{t,f_1(k)f_2(k)}\right) \right| \\
= \; & \mathcal{O}_{a.s.}\left(N^{\beta_2 - 1/2} |\triangle| + N^{\beta_2 - 1}\right)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \max_{1 \le \ell \le p} \left| (nN)^{-1} \sum_{t=1}^{n} \sum_{k=1}^{N} B_\ell(\boldsymbol{x}_{f_1(k)f_2(k)}) \sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right)\left(\varepsilon_{t,f_1(k)f_2(k)} - Z_{t,f_1(k)f_2(k),\varepsilon}\right) \right| \\
& = \mathcal{O}_{a.s}\left(N^{\beta_2 - 1/2} |\triangle| + N^{\beta_2 - 1}\right).
\end{aligned}
$$

Apply the triangle inequality and the result in Lemma A.12, then

$$
\max_{1 \le \ell \le p} \left| (nN)^{-1} \sum_{t=1}^{n} \sum_{k=1}^{N} B_{\ell,p}(\boldsymbol{x}_{f_1(k)f_2(k)}) \sigma\left(\boldsymbol{x}_{f_1(k)f_2(k)}\right) \varepsilon_{t,f_1(k)f_2(k)} \right|
$$

$$= \mathcal{O}_{a.s}\left(n^{-1/2}N^{-1/2}|\triangle|\log^{1/2}N + N^{\beta_2-1/2}|\triangle| + N^{\beta_2-1}\right)$$

It is clear that $(nN)^{-1}\mathbf{X}^\top\sum_{t=1}^n \mathbf{e}_t = \left\{(nN)^{-1}\sum_{t=1}^n\sum_{i=1}^M\sum_{j=1}^{N_i} B_\ell(\boldsymbol{x}_{ij})\,\sigma(\boldsymbol{x}_{ij})\,\varepsilon_{t,ij}\right\}_{\ell=1}^p$, then

$$\left\|(nN)^{-1}\mathbf{X}^\top\sum_{t=1}^n \mathbf{e}_t\right\|_\infty = \mathcal{O}_{a.s}\left(n^{-1/2}N^{-1/2}|\triangle|\log^{1/2}N + N^{\beta_2-1/2}|\triangle| + N^{\beta_2-1}\right).$$

By recalling the definition of $\widetilde{\boldsymbol{e}}_t(\boldsymbol{x})$ and Lemma A.6, one obtains

$$\begin{aligned}\sup_{\boldsymbol{x}\in\Omega} n^{-1}\left|\sum_{t=1}^n \widetilde{\boldsymbol{e}}_t(\boldsymbol{x})\right| &= \left\|n^{-1}N^{-1}\mathbf{B}(\boldsymbol{x})^\top\mathbf{V}\mathbf{X}^\top\sum_{t=1}^n \boldsymbol{e}_t\right\|_\infty \\ &= \mathcal{O}_{a.s}\left(n^{-1/2}N^{-1/2}|\triangle|^{-1}\log^{1/2}N + N^{\beta_2-1/2}|\triangle|^{-1} + N^{\beta_2-1}|\triangle|^{-2}\right)\end{aligned}$$

The proof is completed.

B. Proof of theorems

B.1 Proof of Theorem 1

Under Lemma A.11, $\mathrm{Cov}\left(Z_{jk,\xi}, Z_{j+h,k,\xi}\right) = \sum_{m=0}^\infty a_{mk}a_{m+h,k},\ 1\le j\le n,\ 1\le h\le n-j$. Then,

$$\mathrm{Var}\left(\overline{Z}_{\cdot k,\xi}\right) = \mathbb{E}\left(n^{-1}\sum_{t=1}^n Z_{tk,\xi}\right)^2 = n^{-2}\left[n + 2\mathbb{E}\left(\sum_{1\le t\le j\le n} Z_{tk,\xi}Z_{jk,\xi}\right)\right]$$

$$= n^{-1} + 2n^{-2} \left\{ (n-1) \sum_{t=0}^{\infty} a_{tk} a_{t+1,k} + (n-2) \sum_{t=0}^{\infty} a_{tk} a_{t+2,k} + \cdots + [n-(n-1)] \sum_{t=0}^{\infty} a_{tk} a_{t+n-1,k} \right\}$$

where $\overline{Z}_{\cdot k,\xi} = \sum_{t=1}^{n} Z_{tk,\xi}/n$. We denote $\widetilde{\varphi}_k(\boldsymbol{x}) = \overline{Z}_{\cdot k,\xi}\phi_k(\boldsymbol{x}), k = 1, \ldots, \infty$ and define $\varphi_n(x) = n^{1/2} G_\varphi(\boldsymbol{x}, \boldsymbol{x})^{-1/2} \sum_{k=1}^{\infty} \widetilde{\varphi}_k(\boldsymbol{x})$. For $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_l \in \Omega$ and $b_1, \cdots, b_l \in \mathbb{R}$,

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{i=1}^{l} b_i \varphi_n(\boldsymbol{x}_i)\right) = \mathrm{Var}\left(n^{1/2} \sum_{i=1}^{l} b_i G_\varphi(\boldsymbol{x}_i, \boldsymbol{x}_i)^{-1/2} \sum_{k=1}^{\infty} \overline{Z}_{\cdot k,\xi} \phi_k(\boldsymbol{x}_i)\right) \\
&= n \sum_{i=1}^{l} b_i^2 G_\varphi(\boldsymbol{x}_i, \boldsymbol{x}_i)^{-1} \sum_{k=1}^{\infty} \phi_k^2(\boldsymbol{x}_i) \mathrm{Var}\left(\overline{Z}_{\cdot k,\xi}\right) + 2n \sum_{1 \le i < j \le l} b_i b_j G_\varphi(\boldsymbol{x}_i, \boldsymbol{x}_i)^{-1/2} G_\varphi(\boldsymbol{x}_j, \boldsymbol{x}_j)^{-1/2} \\
&\quad \sum_{k=1}^{\infty} \phi_k(\boldsymbol{x}_i) \phi_k(\boldsymbol{x}_j) \mathrm{Var}\left(\overline{Z}_{\cdot k,\xi}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{Var}\left(\sum_{i=1}^{l} b_i \varphi_n(\boldsymbol{x}_i)\right) \\
&= \sum_{i=1}^{l} b_i^2 + 2 \sum_{1 \le i < j \le l} b_i b_j G_\varphi(\boldsymbol{x}_i, \boldsymbol{x}_i)^{-1/2} G_\varphi(\boldsymbol{x}_j, \boldsymbol{x}_j)^{-1/2} \sum_{k=1}^{\infty} \phi_k(\boldsymbol{x}_i) \phi_k(\boldsymbol{x}_j) \left\{ 1 + 2 \sum_{m=0}^{\infty} \sum_{m'=m+1}^{\infty} a_{mk} a_{m'k} \right\} \\
&= \sum_{i=1}^{l} b_i^2 + 2 \sum_{1 \le i < j \le l} b_i b_j G_\varphi(\boldsymbol{x}_i, \boldsymbol{x}_i)^{-1/2} G_\varphi(\boldsymbol{x}_j, \boldsymbol{x}_j)^{-1/2} G_\varphi(\boldsymbol{x}_i, \boldsymbol{x}_j) \\
&= \mathrm{Var}\left(\sum_{i=1}^{l} b_i \varphi(\boldsymbol{x}_i)\right),
\end{aligned}$$

hence

$$\{\varphi_n(\boldsymbol{x}_1), \cdots, \varphi_n(\boldsymbol{x}_l)\} \to_D \{\varphi(\boldsymbol{x}_1), \cdots, \varphi(\boldsymbol{x}_l)\}. \tag{S.10}$$

Under Assumption (A4), there exists $C_G > 0$ such that $G(\boldsymbol{x}, \boldsymbol{x}) \geq C_G$, $\boldsymbol{x} \in \Omega$. Denote $\omega(\varphi_n, \delta) = \sup_{\boldsymbol{x}, \boldsymbol{x}' \in \Omega, |\boldsymbol{x} - \boldsymbol{x}'| \leq \delta} |\varphi_n(\boldsymbol{x}) - \varphi_n(\boldsymbol{x}')|$. Given the partition $\Omega = \cup_{i=1}^{k} T_i$ that $|T_i| < \delta$, there exists $\sup_{1 \leq i \leq k, \boldsymbol{x}, \boldsymbol{x}' \in T_i,} |\varphi_n(\boldsymbol{x}) - \varphi_n(\boldsymbol{x}')| \leq \omega(\varphi_n, \delta)$. The definition of $\omega(\varphi_n, \delta)$ implies that

$$
\begin{aligned}
\omega(\varphi_n, \delta) &= \sup_{\boldsymbol{x}, \boldsymbol{x}' \in \Omega, |\boldsymbol{x} - \boldsymbol{x}'| \leq \delta} |\varphi_n(\boldsymbol{x}) - \varphi_n(\boldsymbol{x}')| \\
&\leq \sup_{\boldsymbol{x}, \boldsymbol{x}' \in \Omega, |\boldsymbol{x} - \boldsymbol{x}'| \leq \delta} n^{1/2} C_{\varphi}^{-1/2} \sum_{k=1}^{\infty} |\phi_k(\boldsymbol{x}) - \phi_k(\boldsymbol{x}')| \left|\overline{Z}_{\cdot k, \xi}\right| \\
&\leq n^{1/2} C_{\varphi}^{-1/2} \delta^{\mu} \sum_{k=1}^{\infty} \|\phi_k\|_{0,\mu} \left|\overline{Z}_{\cdot k, \xi}\right|.
\end{aligned}
$$

Since $\mathbb{E}\left|\overline{Z}_{\cdot k, \xi}\right| = (2/\pi)^{1/2} \mathrm{Var}\left(\overline{Z}_{\cdot k, \xi}\right)^{1/2}$, thus

$$
\begin{aligned}
\mathbb{P}\left[\omega(\varphi_n, \delta) \geq \epsilon\right] &\leq \mathbb{P}\left(n^{1/2} \delta^{\mu} C_{\varphi}^{-1/2} \sum_{k=1}^{\infty} \|\phi_k\|_{0,\mu} \left|\overline{Z}_{\cdot k, \xi}\right| \geq \epsilon\right) \\
&\leq \mathbb{P}\left(\sum_{k=1}^{\infty} \|\phi_k\|_{0,\mu} \left|\overline{Z}_{\cdot k, \xi}\right| \geq \frac{\epsilon}{n^{1/2} \delta^{\mu} C_{\varphi}^{-1/2}}\right) \\
&\leq \frac{n^{1/2} \delta^{\mu} C_{\varphi}^{-1/2} \sum_{k=1}^{\infty} \|\phi_k\|_{0,\mu} \mathbb{E}\left|\overline{Z}_{\cdot k, \xi}\right|}{\epsilon} \\
&\leq \frac{(2/\pi)^{1/2} \delta^{\mu} C_{\varphi}^{-1/2} \sum_{k=1}^{\infty} \|\phi_k\|_{0,\mu} \left\{n \mathrm{Var}\left(\overline{Z}_{\cdot k, \xi}\right)\right\}^{1/2}}{\epsilon}.
\end{aligned}
$$

Note that $\sum_{k=1}^{\infty} \|\phi_k\|_{0,\mu} < +\infty$ under Assumption (A4) and $n\mathrm{Var}\left(\overline{Z}_{\cdot k,\xi}\right) \to 1 + 2\sum_{t=0}^{\infty}\sum_{t'=t+1}^{\infty} a_{tk}a_{t'k}$ as $n \to \infty$, it is clear that

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \frac{(2/\pi)^{1/2}\delta^{\mu}C_{\varphi}^{-1/2}\sum_{k=1}^{\infty}\|\phi_k\|_{0,\mu}\left\{n\mathrm{Var}\left(\overline{Z}_{\cdot k,\xi}\right)\right\}^{1/2}}{\epsilon} = 0,$$

thus equation (S.2) is satisfied. According to (S.10) and Lemma A.4, $\varphi_n \to_D \varphi$.

Denote $\overline{\xi}_{\cdot k} = \sum_{t=1}^{n} \xi_{tk}$, and note that

$$\begin{aligned} & n^{1/2}\sup_{\boldsymbol{x}\in\Omega} G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}\right)^{-1/2}\left|\sum_{k=1}^{\infty}\left(\overline{Z}_{\cdot k,\xi}-\overline{\xi}_{\cdot k}\right)\phi_k(\boldsymbol{x})\right| \\ &\le n^{1/2}\sup_{\boldsymbol{x}\in\Omega} G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}\right)^{-1/2}\sum_{k=1}^{k_n}\left|\overline{Z}_{\cdot k,\xi}-\overline{\xi}_{\cdot k}\right|\left|\phi_k(\boldsymbol{x})\right| \\ &\quad + n^{1/2}\sup_{\boldsymbol{x}\in\Omega} G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}\right)^{-1/2}\sum_{k=k_n+1}^{\infty}\left|\overline{Z}_{\cdot k,\xi}-\overline{\xi}_{\cdot k}\right|\left|\phi_k(\boldsymbol{x})\right|. \end{aligned}$$

According to (S.7), there exists

$$\mathbb{P}\left\{\max_{1\le k\le k_n}\left|\overline{\xi}_{\cdot k}-\overline{Z}_{\cdot k,\xi}\right| > C_3 n^{\beta_3-1}\right\} < C_4 n^{-\gamma_3}.$$

By Borel Cantelli lemma, one has

$$\max_{1\le k\le k_n}\left|\overline{\xi}_{\cdot k}-\overline{Z}_{\cdot k,\xi}\right| = \mathcal{O}_{a.s.}\left(n^{\beta_3-1}\right). \tag{S.11}$$

By Assumption (A4), $\sum_{k=1}^{\infty} \|\phi_k\|_\infty < +\infty$, thus $\sum_{k=1}^{k_n} \|\phi_k\|_\infty < C$ for some constant C. Together with (S.11) and Assumption (A3), one obtains that

$$\begin{aligned}
& n^{1/2} \sup_{\boldsymbol{x}\in\Omega} G_\varphi(\boldsymbol{x},\boldsymbol{x})^{-1/2} \sum_{k=1}^{k_n} \left|\overline{Z}_{\cdot k,\xi} - \overline{\xi}_{\cdot k}\right| |\phi_k(\boldsymbol{x})| \\
& \leq n^{1/2} C_G^{-1/2} \sup_{\boldsymbol{x}\in\Omega} \sum_{k=1}^{k_n} |\phi_k(\boldsymbol{x})| \max_{1\leq k\leq k_n} \left|\overline{\xi}_{\cdot k} - \overline{Z}_{\cdot k,\xi}\right| \\
& \leq n^{1/2} C_G^{-1/2} \sum_{k=1}^{k_n} \|\phi_k\|_{\infty,\Omega} \max_{1\leq k\leq k_n} \left|\overline{\xi}_{\cdot k} - \overline{Z}_{\cdot k,\xi}\right| \\
& \leq n^{1/2} C_G^{-1/2} C \mathcal{O}_{a.s.}\left(n^{\beta_3-1}\right) = \mathcal{O}_{a.s.}\left(n^{\beta_3-1/2}\right) = o_{a.s.}(1)
\end{aligned} \tag{S.12}$$

Note that

$$\left(\mathbb{E}\left|\overline{\xi}_{\cdot k}\right|\right)^2 = \left(\mathbb{E}\left|\overline{Z}_{\cdot k,\xi}\right|\right)^2 \leq \mathbb{E}\overline{Z}^2_{\cdot k,\xi} = n^{-1} + 2n^{-2}\left\{\sum_{m=1}^{n-1}\sum_{t=0}^{\infty}(n-m)a_{tk}a_{t+m,k}\right\},$$

thus $\mathbb{E}\left|\overline{\xi}_{\cdot k}\right| = \mathbb{E}\left|\overline{Z}_{\cdot k,\xi}\right| = \mathcal{O}\left(n^{-1/2}\right)$. In addition, Assumption (A4) states that $\sum_{k=k_n+1}^{\infty} \|\phi_k\|_\infty = o\left(n^{-1/2}\right)$, then there exists

$$\begin{aligned}
& \mathbb{E} n^{1/2} \sup_{\boldsymbol{x}\in\Omega} G_\varphi(\boldsymbol{x},\boldsymbol{x})^{-1/2} \sum_{k=k_n+1}^{\infty} \left|\overline{Z}_{\cdot k,\xi} - \overline{\xi}_{\cdot k}\right| |\phi_k(x)| \\
& \leq n^{1/2} C_G^{-1/2} \sum_{k=k_n+1}^{\infty} \|\phi_k\|_{\infty,\Omega} \mathbb{E}\left|\overline{Z}_{\cdot k,\xi} - \overline{\xi}_{\cdot k}\right| \\
& \leq n^{1/2} C_G^{-1/2} o\left(n^{-1/2}\right) \mathcal{O}\left(n^{-1/2}\right) = o(1)
\end{aligned} \tag{S.13}$$

Combining (S.12) and (S.13), one has

$$\mathbb{E}n^{1/2}\sup_{\boldsymbol{x}\in\Omega} G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}\right)^{-1/2}\left|\sum_{k=1}^{\infty}\left(\overline{Z}_{\cdot k,\xi}-\overline{\xi}_{\cdot k}\right)\phi_k(\boldsymbol{x})\right| = \mathcal{O}\left(1\right),$$

hence

$$n^{1/2}\sup_{\boldsymbol{x}\in\Omega} G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}\right)^{-1/2}\left|\sum_{k=1}^{\infty}\left(\overline{Z}_{\cdot k,\xi}-\overline{\xi}_{\cdot k}\right)\phi_k(\boldsymbol{x})\right| = \mathcal{O}_p\left(1\right).$$

Note that

$$\varphi_n(\boldsymbol{x}) - n^{1/2}G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}\right)^{-1/2}\left\{\overline{m}\left(\boldsymbol{x}\right)-m\left(\boldsymbol{x}\right)\right\} = n^{1/2}G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}\right)^{-1/2}\sum_{k=1}^{\infty}\left(\overline{Z}_{\cdot k,\xi}-\overline{\xi}_{\cdot k}\right)\phi_k(\boldsymbol{x}),$$

hence

$$\sup_{\boldsymbol{x}\in\Omega}\left|\varphi_n(\boldsymbol{x}) - n^{1/2}G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}\right)^{-1/2}\left\{\overline{m}\left(\boldsymbol{x}\right)-m\left(\boldsymbol{x}\right)\right\}\right| = \mathcal{O}_p\left(1\right).$$

The proof is completed by applying Slutsky's theorem.

B.2 Proof of Theorem 2

For any $k = 1\ldots,\infty$, let $\widetilde{\phi}_k(\boldsymbol{x}) = \widetilde{\mathbf{B}}(\boldsymbol{x})(\widetilde{\mathbf{X}}^{\top}\widetilde{\mathbf{X}})^{-1}\widetilde{\mathbf{X}}\boldsymbol{\phi}_k$. According to the equation (S.1),

$$\widehat{\eta}_t(\boldsymbol{x}) - \eta_t(\boldsymbol{x}) = \widetilde{m}(\boldsymbol{x}) - m(\boldsymbol{x}) + \widetilde{\xi}_t(\boldsymbol{x}) - \xi_t(\boldsymbol{x}) + \widetilde{e}_t(\boldsymbol{x}) \tag{S.9}$$

By Lemma A.8, for any $k = 1\dots,\infty$, there exist a constant $C_{d,r}$, independent of m and ϕ_k, such that

$$\begin{aligned} \left\|\widetilde{m}-m\right\|_{\infty,\Omega} &\leq C_{d,r}\left|m\right|_{d+1,\Omega,\infty}\left|\triangle\right|^{d+1}, \\ \left\|\widetilde{\phi}_k-\phi_k\right\|_{\infty,\Omega} &\leq C_{d,r}\left|\phi_k\right|_{d+1,\Omega,\infty}\left|\triangle\right|^{d+1}, \end{aligned}$$

which implies that

$$\left\|\widetilde{\xi}_t-\xi_t\right\|_{\infty,\Omega} = \sum_{k=1}^{\infty}\left|\xi_{tk}\right|\left\|\widetilde{\phi}_k-\phi_k\right\|_{\infty,\Omega} \leq C_{d,r}W_t|\triangle|^{d+1},$$

where $W_t = \sum_{k=1}^{\infty}\left|\xi_{tk}\right|\left|\phi_k\right|_{d+1,\Omega,\infty}, t = 1,\dots,n$ are identically distributed nonnegative random variables with r_1-th finite absolute moment under the Assumptions (A4) and (A5). Hence

$$\mathbb{P}\left\{\max_{1\leq t\leq n}W_i > (n\log n)^{2/r_1}\right\} \leq n\frac{\mathbb{E}W_t^{r_1}}{(n\log n)^2} = \mathbb{E}W_t^{r_1}n^{-1}(\log n)^{-2},$$

and the Borel-Canteill lemma ensures that $\max_{1\leq t\leq n}W_i = \mathcal{O}_{a.s}\left\{(n\log n)^{2/r_1}\right\}$. Together with equation (S.9) and Lemma A.13, one can obtain that

$$\sup_{\boldsymbol{x}\in\Omega}\left|\widehat{m}(\boldsymbol{x})-\overline{m}(\boldsymbol{x})\right| = \sup_{\boldsymbol{x}\in\Omega}\left|n^{-1}\sum_{t=1}^{n}\left(\widehat{\eta}_t(\boldsymbol{x})-\eta_t(\boldsymbol{x})\right)\right|$$

$$
\begin{aligned}
&\le \max_{1\le t\le n}\sup_{\boldsymbol{x}\in\Omega}\left|\widetilde{\xi}_t(\boldsymbol{x})-\xi_t(\boldsymbol{x})\right|+\sup_{\boldsymbol{x}\in\Omega}|\widetilde{m}(\boldsymbol{x})-m(\boldsymbol{x})|+\max_{1\le t\le n}\sup_{\boldsymbol{x}\in\Omega}\left|n^{-1}\sum_{t=1}^{n}\widetilde{e}_t(\boldsymbol{x})\right| \\
&= \mathcal{O}_{a.s.}\left(|\triangle|^{d+1}(n\log n)^{2/r_1}+|\triangle|^{d+1}+N^{\beta_2-1/2}|\triangle|^{-1}+n^{-1/2}N^{-1/2}|\triangle|^{-1}\log^{1/2}N\right) \\
&= \mathcal{O}_{a.s.}\left(|\triangle|^{d+1}(n\log n)^{2/r_1}+N^{\beta_2-1/2}|\triangle|^{-1}\right)=o_p\left(n^{-1/2}\right).
\end{aligned}
$$

B.3 Proof of Theorem 3

From Assumption (A5), one obtains that $\rho_a < -2$, satisfying the assumptions of Theorem 3 in Wu (2005). Together with Assumption (A5′), one derives that there exists i.i.d. normal sequences $Z_{tk,\xi}$, $t = 1, \ldots, n$, $k = 1, \ldots, \kappa_n$, such that

$$
\max_{1\le k\le\kappa_n}\left|\sum_{t=1}^{n}\xi_{tk}-\sqrt{\lambda_k^*}\sum_{t=1}^{n}Z_{tk,\xi}\right|=\mathcal{O}_{a.s.}\left(n^{1/4+\alpha}(\log n)^{3/4}(\log\log n)^{1/2}\right). \tag{S.13}
$$

Note that $\alpha < 1/4$, thus (S.11) - (S.13) still hold. One can complete the poof following the same arguments in the proof of Theorem 1 and 2.

B.4 Proof of Theorem 4

Define the B-approximate process as

$$
\begin{aligned}
\widetilde{\xi}_t(\cdot) &= \mathbb{E}\left\{\xi_t(\cdot)\,|\zeta_{ik}, i\ge t-B, k\ge 1\right\} \\
&= \sum_{k=1}^{\infty}\widetilde{\xi}_{tk,B}\phi_k(\cdot)=\sum_{k=1}^{\infty}\left(\sum_{t'=0}^{B}a_{t',k}\zeta_{t-t',k}\right)\phi_k(\cdot).
\end{aligned}
$$

Let $\delta_j(\boldsymbol{x}) = B^{-1}\sum_{t=B(j-1)+1}^{Bj}\{\eta_t(\boldsymbol{x}) - \overline{m}(\boldsymbol{x})\}$ and $\widetilde{\delta}_j(\boldsymbol{x}) = B^{-1}\sum_{t=B(j-1)+1}^{Bj}\widetilde{\xi}_t(\boldsymbol{x})$. Define the corresponding covariance functions

$$
\begin{aligned}
G_{\varphi,B}(\boldsymbol{x},\boldsymbol{x}') &= \sum_{k=1}^{\infty}\phi_k(\boldsymbol{x})\phi_k(\boldsymbol{x}')\left\{1+2\sum_{t=0}^{B}(1-t/B)\sum_{t'=t+1}^{\infty}a_{tk}a_{t'k}\right\},\\
\widetilde{G}_{\varphi}(\boldsymbol{x},\boldsymbol{x}') &= \frac{B}{l}\sum_{j=1}^{l}\left\{\delta_j(\boldsymbol{x})\,\delta_j(\boldsymbol{x}') - \overline{\delta}(\boldsymbol{x})\,\overline{\delta}(\boldsymbol{x}')\right\},\\
\widetilde{G}_{\varphi,B}(\boldsymbol{x},\boldsymbol{x}') &= \frac{B}{l}\sum_{j=1}^{l}\left\{\widetilde{\delta}_j(\boldsymbol{x})\,\widetilde{\delta}_j(\boldsymbol{x}') - \overline{\widetilde{\delta}}(\boldsymbol{x})\,\overline{\widetilde{\delta}}(\boldsymbol{x}')\right\},
\end{aligned}
$$

in which $G_{\varphi,B}(\boldsymbol{x},\boldsymbol{x}') = B\mathrm{Cov}(\widetilde{\delta}_j(\boldsymbol{x}),\widetilde{\delta}_j(\boldsymbol{x}'))$, $\overline{\delta}(\boldsymbol{x}) = l^{-1}\sum_{j=1}^{l}\delta_j(\boldsymbol{x})$ and $\overline{\widetilde{\delta}}(\boldsymbol{x}) = l^{-1}\sum_{j=1}^{l}\widetilde{\delta}_j(\boldsymbol{x})$, while $\widetilde{G}_{\varphi}(\boldsymbol{x},\boldsymbol{x}')$ and $\widetilde{G}_{\varphi,B}(\boldsymbol{x},\boldsymbol{x}')$ are block estimators of $G_{\varphi}(\boldsymbol{x},\boldsymbol{x}')$ by infeasible trajectories and B-approximate trajectories respectively.

We decompose the difference between $G_{\varphi}(\boldsymbol{x},\boldsymbol{x}')$ and $\widehat{G}_{\varphi}(\boldsymbol{x},\boldsymbol{x}')$ into the following four terms:

$$
\begin{aligned}
&\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|G_{\varphi}(\boldsymbol{x},\boldsymbol{x}') - \widehat{G}_{\varphi}(\boldsymbol{x},\boldsymbol{x}')\right|\\
\leq\;&\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widehat{G}_{\varphi}(\boldsymbol{x},\boldsymbol{x}') - \widetilde{G}_{\varphi}(\boldsymbol{x},\boldsymbol{x}')\right| + \sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widetilde{G}_{\varphi}(\boldsymbol{x},\boldsymbol{x}') - \widetilde{G}_{\varphi,B}(\boldsymbol{x},\boldsymbol{x}')\right|\\
&+\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widetilde{G}_{\varphi,B}(\boldsymbol{x},\boldsymbol{x}') - G_{\varphi,B}(\boldsymbol{x},\boldsymbol{x}')\right| + \sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|G_{\varphi,B}(\boldsymbol{x},\boldsymbol{x}') - G_{\varphi}(\boldsymbol{x},\boldsymbol{x}')\right|.
\end{aligned}
$$

Lemma A.14. *Under Assumptions (A1)–(A6), there exists*

$$\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widehat{G}_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}'\right)-\widetilde{G}_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}'\right)\right|=\mathcal{O}_p\left(B\left(n\log n\right)^{4/r_1}|\triangle|^{d+1}+B\left(n\log n\right)^{2/r_1}N^{\beta_2-1/2}|\triangle|^{-1}\right).$$

Proof. Note that

$$\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widehat{G}_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}'\right)-\widetilde{G}_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}'\right)\right|\leq 2B\max_{1\leq j\leq l}\left(\|\delta_j\|_\infty+\left\|\widehat{\delta}_j\right\|_\infty\right)\max_{1\leq j\leq j}\left\|\delta_j-\widehat{\delta}_j\right\|_\infty,$$

hence

$$\begin{aligned}\max_{1\leq j\leq l}\left\|\delta_j-\widehat{\delta}_j\right\|_\infty &= \max_{1\leq t\leq n}\|\eta_t-\widehat{\eta}_t\|_\infty+\mathcal{O}_p(n^{-1/2}),\\ \max_{1\leq j\leq l}\left\|\widehat{\delta}_j\right\|_\infty &\leq \max_{1\leq j\leq l}\|\delta_j\|_\infty+\max_{1\leq j\leq l}\left\|\delta_j-\widehat{\delta}_j\right\|_\infty.\end{aligned}$$

Similar with the proof of Theorem 2, one can get

$$\max_{1\leq j\leq l}\|\delta_j\|_\infty\leq\max_{1\leq t\leq n}\|\xi_t\|_\infty=\mathcal{O}_p((n\log n)^{2/r_1}).$$

The proof is completed. □

Lemma A.15. *Under Assumptions (A1)–(A6), one has*

$$\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widetilde{G}_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}'\right)-\widetilde{G}_{\varphi,B}\left(\boldsymbol{x},\boldsymbol{x}'\right)\right|=\mathcal{O}_p\left(B^{\rho_a+3/2}l\log l\left(n\log n\right)^{2/r_1}\right).$$

Proof. Following the analogous steps of the previous lemma, one has

$$\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widetilde{G}_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}'\right)-\widetilde{G}_{\varphi,B}\left(\boldsymbol{x},\boldsymbol{x}'\right)\right| \leq 2B\max_{1\leq j\leq l}\|\delta_j\|_{\infty}\max_{1\leq j\leq l}\left\|\delta_j-\widetilde{\delta}_j\right\|_{\infty}.$$

Note that

$$\begin{aligned}\mathbb{E}\left\|\delta_j-\widetilde{\delta}_j\right\|_{\infty}^2 &\leq \sum_{k=1}^{\infty}\mathbb{E}\left|\sum_{t'=B+1}^{\infty}a_{t'k}\zeta_{t-t',k}\right|^2\|\phi_k\|_{\infty}^2\\ &\leq \sum_{k=1}^{\infty}\sum_{t'=B+1}^{\infty}(B+1)^{2\rho_a}\|\phi_k\|_{\infty}^2\\ &= \mathcal{O}(B^{2\rho_a+1}),\end{aligned}$$

thus

$$\mathbb{P}\left(\max_{1\leq l\leq j}\left\|\delta_j-\widetilde{\delta}_j\right\|_{\infty}>B^{\rho_a+1/2}l\log l\right) \leq \frac{1}{l\log^2 l}.$$

Borel-Cantelli lemma leads to that $\max_{1\leq l\leq j}\left\|\delta_j-\widetilde{\delta}_j\right\|_{\infty}=\mathcal{O}\left(B^{\rho_a+1/2}l\log l\right)$. Combining with $\max_{1\leq j\leq l}\|\delta_j\|_{\infty}=\mathcal{O}_p((n\log n)^{2/r_1})$, the lemma holds consequently. □

Lemma A.16. *Under Assumptions (A1)–(A6), one has*

$$\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widetilde{G}_{\varphi,B}\left(\boldsymbol{x},\boldsymbol{x}'\right)-G_{\varphi,B}\left(\boldsymbol{x},\boldsymbol{x}'\right)\right|=\mathcal{O}_p(Bl^{-1/2}).$$

Proof. Denote $\delta_{jk} = B^{-1}\sum_{t=B(j-1)+1}^{Bj}\sum_{t'=0}^{B} a_{t',k}\zeta_{t-t',k}$, then

$$\begin{aligned}\left|\widetilde{G}_{\varphi,B}\left(\boldsymbol{x},\boldsymbol{x}'\right)-G_{\varphi,B}\left(\boldsymbol{x},\boldsymbol{x}'\right)\right| &\leq \left|\sum_{k=1}^{\infty}\left(B\delta_{\cdot,kk}-\widetilde{\lambda}_{k,B}\right)\phi_k(\boldsymbol{x})\phi_k(\boldsymbol{x}')\right| \\ &\quad +2\left|\sum_{k<k'} B\delta_{\cdot,kk'}\phi_k(\boldsymbol{x})\phi_k(\boldsymbol{x}')\right|,\end{aligned}$$

where $\delta_{\cdot,kk'} = l^{-1}\sum_{j=1}^{l}\delta_{jk}\delta_{jk'}$ and $\widetilde{\lambda}_{k,B} = 1 + 2\sum_{t=1}^{B}(1-t/B)\sum_{t'=t}^{\infty} a_{tk}a_{t'k}$.

Given the independence of $\delta_{ik}, \delta_{jk}, |i-j|>1$, it follows that

$$\begin{aligned}\mathbb{E}\left(B\delta_{\cdot,kk}\right) &= \mathbb{E}\left(B\delta_{jk}^2\right) = \widetilde{\lambda}_{k,B}, \\ \mathbb{E}\left(B\delta_{\cdot,kk}-\widetilde{\lambda}_{k,B}\right)^2 &= l^{-2}\mathbb{E}\left\{B^2\sum_{j=1}^{l}\delta_{jk}^4 + 2\sum_{i<j}^{l}B^2\delta_{ik}^2\delta_{jk}^2\right\} - \widetilde{\lambda}_{k,B}^2 \\ &= l^{-1}B^2\left\{\mathbb{E}\delta_{jk}^4 + \frac{3-2l}{l}\widetilde{\lambda}_{k,B}^2 + \frac{l-1}{l}\mathbb{E}\delta_{jk}^2\delta_{j+1k}^2\right\} \\ &= \mathcal{O}_p\left(l^{-1}B^2\right).\end{aligned}$$

Based on Assumption (A5′) , one can obtain that

$$\begin{aligned}\mathbb{E}\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\sum_{k=1}^{\infty}\left(B\delta_{\cdot,kk}-\widetilde{\lambda}_{k,B}\right)\phi_k(\boldsymbol{x})\phi_k(\boldsymbol{x}')\right| &\leq \sum_{k=1}^{\infty}\mathbb{E}\left|B\delta_{\cdot,kk}-\widetilde{\lambda}_{k,B}\right|\|\phi_k\|^2 \\ &= \mathcal{O}_p\left(Bl^{-1/2}\right).\end{aligned}$$

Similarly, one can derive the order of the second part. Thus the proof is completed.

□

Lemma A.17. *There exists*

$$\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|G_{\varphi,B}\left(\boldsymbol{x},\boldsymbol{x}'\right)-G_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}'\right)\right|=\mathcal{O}\left(B^{2\rho_a+2}\right).$$

Proof. Denote by $\widetilde{\lambda}_k = 1+2\sum_{t=0}^{\infty}\sum_{t'=t+1}^{\infty}a_{tk}a_{t'k}$, then

$$\begin{aligned}\sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\widetilde{G}_{\varphi,B}\left(\boldsymbol{x},\boldsymbol{x}'\right)-\widetilde{G}_{\varphi}\left(\boldsymbol{x},\boldsymbol{x}'\right)\right| &= \sup_{\boldsymbol{x},\boldsymbol{x}'\in\Omega}\left|\sum_{k=1}^{\infty}\left(\widetilde{\lambda}_{k,B}-\widetilde{\lambda}_k\right)\phi_k(\boldsymbol{x})\phi_k(\boldsymbol{x}')\right| \\ &\leq \sum_{k=1}^{\infty}\left|\widetilde{\lambda}_{k,B}-\widetilde{\lambda}_k\right|\|\phi_k\|^2.\end{aligned}$$

Simple algebra leads to

$$\begin{aligned}\left|\widetilde{\lambda}_{k,B}-\widetilde{\lambda}_k\right| &= 2\left|B^{-1}\sum_{t=0}^{B-1}t\sum_{t'=t+1}^{\infty}a_{tk}a_{t'k}+\sum_{t=B}^{\infty}\sum_{t'=t+1}^{\infty}a_{tk}a_{t'k}\right| \\ &\leq 2CB^{-1}\sum_{t=0}^{B-1}|a_{tk}|\,t(t+1)^{\rho_a+1}+2CB^{2\rho_a+2} \\ &\leq 2CB^{-1}\sum_{t=0}^{B-1}(t+1)^{2\rho_a+2}+2CB^{2\rho_a+2} \\ &= \mathcal{O}\left(B^{2\rho_a+2}\right)=o\left(1\right).\end{aligned}$$

Thus the lemma holds. □

To ensure the weak convergence, we need to determine the order of B and l.

Assume $B \asymp n^m$, $l \asymp n^{1-m}$, together with $n = \mathcal{O}\left(N^{\theta}\right)$, $|\triangle| \asymp N^{\gamma} d_N$, then m should satisfy the following inequalities:

$$\begin{cases} (m + 4/r_1)\theta - (d+1)\gamma < 0 \\ (m + 2/r_1)\theta + \beta_2 - 1/2 + \gamma < 0 \\ (\rho_a - 1)m + 2/r_1 + 1 < 0 \\ m < 1/4 \end{cases}$$

Considering the defaults of θ, γ and d in Remark 1, it is easy to find that $m = 1/5$ is a reasonable choice which can guarantee the above inequalities hold. Thus we specify $B \asymp n^{1/5}$ in the implementation.

C. Additional simulation

C1. Dependence on the block size B

The block size B is involved in estimating the limiting covariance function $G_{\varphi}(\boldsymbol{x}, \boldsymbol{x}')$, thus representing the order of the FMA(∞) to some extent. In order to investigate its influence, we compare the performance of SCCs constructed in 3 different block sizes: $B = \left[n^{1/5} \log\log n\right]$ (the one we recommend in Section 4), $B = 1$ (ignore the dependence and view as independent blocks) and $B = 3\left[n^{1/5} \log\log n\right]$ (overestimate the dependence/FMA(∞) order).

The data is generated from (5.1) with homoscedastic errors $\sigma(s,t) \equiv 0.1$, and ε

obeys standard normal distribution. Table 1 displays the coverage frequencies over 500 replications of the various SCCs. It is clear that the independent SCC ($B = 1$) severely suffers from the problem of dependence misspecification, leading to poor coverage frequency. While overestimating the dependence have slighter influence, but the empirical coverage rate of its SCC still can not approach the nominal level.

Table 1: Coverage frequencies from 500 replications based on homoscedastic errors $\sigma(s,t) \equiv 0.1$ with different block sizes.

Domain Ω	N	10000			20000		
	Block size B	$B=[n^{1/5}\log\log n]$	$B=1$	$B=3[n^{1/5}\log\log n]$	$B=[n^{1/5}\log\log n]$	$B=1$	$B=3[n^{1/5}\log\log n]$
Square	$\alpha=0.10$	0.858	0.614	0.802	0.894	0.568	0.834
	$\alpha=0.05$	0.916	0.720	0.876	0.942	0.684	0.930
	$\alpha=0.025$	0.954	0.820	0.924	0.974	0.768	0.962
	$\alpha=0.01$	0.984	0.898	0.962	0.994	0.874	0.986
Regular 12 polygon	$\alpha=0.10$	0.738	0.594	0.782	0.894	0.638	0.790
	$\alpha=0.05$	0.858	0.722	0.848	0.952	0.762	0.890
	$\alpha=0.025$	0.926	0.806	0.928	0.976	0.848	0.938
	$\alpha=0.01$	0.980	0.890	0.968	0.994	0.908	0.976
Regular 12 polygon with a square hole	$\alpha=0.10$	0.838	0.582	0.834	0.904	0.640	0.868
	$\alpha=0.05$	0.928	0.728	0.908	0.95	0.754	0.950
	$\alpha=0.025$	0.972	0.810	0.948	0.976	0.846	0.974
	$\alpha=0.01$	0.984	0.896	0.972	0.984	0.902	0.986

C2. Simulation for nonlinear processes

Next we investigate the performance of the proposed SCC under the nonlinear processes setting. The data is generated from the following model:

$$Y_{t,ij} = m(\boldsymbol{x}_{ij}) + \sum_{k=1}^{7} \xi_{tk}\phi_k(\boldsymbol{x}_{ij}) + \sigma(\boldsymbol{x}_{ij})\varepsilon_{t,ij}, \quad t = 1, \ldots, n.$$

The FPC score ξ_{tk} is generated from two different nonlinear time series. For $t = 1, \ldots, n$, $k = 1, \ldots 7$,

$$\xi_{tk} = 0.2\zeta_{t,k} \left|\zeta_{t,k}\right| + 0.3\zeta_{t-1,k} \left|\zeta_{t-1,k}\right| + 0.5\zeta_{t-2,k} + 0.6\zeta_{t-1,k} \tag{S.14}$$

$$\xi_{tk} = 0.3\sqrt{2}\zeta_{t,k}\zeta_{t-1,k} + 0.4\sqrt{2}\zeta_{t-2,k}\zeta_{t-3,k} + 0.5\zeta_{t-4,k}\zeta_{t-5,k} + 0.5\zeta_{t-6,k}, \tag{S.15}$$

where $\zeta_{t,k}$ are standard normal variables, and for (S.14), one may divide it by its variance for unit variance. The mean function $m(\cdot)$, eigenfunctions $\phi_k(\cdot)$, domain Ω and distribution of ϵ are the same as those in Section 5. We choose homoscedastic errors $\sigma(s,t) \equiv 0.1$.

Tables 2 and 3 establishes display the empirical coverage rate of the 500 replications of the true mean function $m(\cdot)$ being covered by the bivariate spline SCCs under nonlinear processes (S.14) and (S.15). It is shown that in both scenarios, as the sample size n increases, the coverage rate of the SCC gets closer to the predetermined confidence level as the sample size increases, which demonstrates the validity of our proposed method for nonlinear processes.

References

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Table 2: Coverage frequencies from 500 replications based on homoscedastic errors $\sigma(s,t) \equiv 0.1$. with (S.14).

Domain Ω	N	10000			20000		
	Distribution of ε	normal	uniform	Laplace	normal	uniform	Laplace
Square	$\alpha = 0.10$	0.890	0.886	0.894	0.905	0.908	0.907
	$\alpha = 0.05$	0.936	0.948	0.940	0.958	0.960	0.955
	$\alpha = 0.025$	0.970	0.982	0.972	0.978	0.977	0.972
	$\alpha = 0.01$	0.986	0.992	0.990	0.988	0.994	0.986
Regular 12 polygon	$\alpha = 0.10$	0.692	0.698	0.692	0.886	0.894	0.898
	$\alpha = 0.05$	0.766	0.782	0.774	0.940	0.944	0.936
	$\alpha = 0.025$	0.880	0.858	0.868	0.974	0.974	0.970
	$\alpha = 0.01$	0.936	0.930	0.936	0.988	0.984	0.984
Regular 12 polygon with a square hole	$\alpha = 0.10$	0.824	0.844	0.828	0.898	0.906	0.890
	$\alpha = 0.05$	0.920	0.924	0.914	0.952	0.950	0.948
	$\alpha = 0.025$	0.956	0.966	0.950	0.974	0.972	0.973
	$\alpha = 0.01$	0.984	0.976	0.976	0.998	0.996	0.992

Table 3: Coverage frequencies from 500 replications based on homoscedastic errors $\sigma(s,t) \equiv 0.1$. with (S.15).

Domain Ω	N	10000			20000		
	Distribution of ε	normal	uniform	Laplace	normal	uniform	Laplace
Square	$\alpha = 0.10$	0.888	0.890	0.892	0.904	0.900	0.903
	$\alpha = 0.05$	0.946	0.950	0.946	0.954	0.956	0.954
	$\alpha = 0.025$	0.978	0.982	0.976	0.974	0.978	0.970
	$\alpha = 0.01$	0.988	0.992	0.990	0.992	0.994	0.994
Regular 12 polygon	$\alpha = 0.10$	0.670	0.656	0.678	0.882	0.884	0.894
	$\alpha = 0.05$	0.748	0.758	0.764	0.948	0.946	0.948
	$\alpha = 0.025$	0.862	0.890	0.854	0.978	0.974	0.972
	$\alpha = 0.01$	0.930	0.946	0.942	0.984	0.990	0.986
Regular 12 polygon with a square hole	$\alpha = 0.10$	0.810	0.820	0.812	0.902	0.900	0.906
	$\alpha = 0.05$	0.898	0.910	0.902	0.952	0.956	0.958
	$\alpha = 0.025$	0.946	0.956	0.946	0.974	0.976	0.972
	$\alpha = 0.01$	0.976	0.984	0.978	0.986	0.990	0.990

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